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MITIGATION OF THREE-DIMENSIONAL VIBRATION OF INCLINED SAG CABLE USING DISCRETE OIL DAMPERS—I. FORMULATION

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Three-dimensional small-amplitude free and forced vibration problems of an inclined sag cable equipped with discrete oil dampers are formulated in this paper using a hybrid method. The hybrid method can readily consider cable sag, cable inclination, cable internal damping, damper direction, damper stiffness and others. Multi-pairs of oil dampers with either symmetric or unsymmetric arrangement can also be dealt with. From complex vibration analyses, the damping ratios of a cable, attributed to the installed oil dampers, in both in-plane and out-of-plane modes of vibration can be estimated. The dynamic in-plane and out-of-plane responses of the cable under harmonic excitation can also be determined in the frequency domain. The application of the derived formulae to sag cables in a real cable-stayed bridge is presented in part II of this paper with extensive parametric studies.

1. INTRODUCTION

In the past few decades, cable-stayed bridges have found wide application throughout the world. The main span of the bridges has reached a level of 900 m, leading to very long stayed cables. These long stayed cables with very low internal structural damping are prone to wind-induced vibration, wind-rain-induced vibration, or vibration due to parametric excitation caused by the motion of either bridge deck or towers [1, 2]. To overcome cable vibration problems, oil dampers have been used to connect the cable to the bridge deck, as done in the Brotonne bridge in France, the Sunshine Skyway bridge in U.S.A., and the Aratus bridge in Japan.

To obtain maximum vibration energy dissipation and to facilitate the design of oil dampers, some research has been carried out on the achievable maximum modal damping ratios in the cable and the optimum parameters for the oil damper [3–7]. In these previous studies, an inclined cable with an oil damper was modelled as a horizontal taut string with the damper normal to the string near its support. Complex eigenvalue analysis through the Galerkin method was used to find solutions. Cable sag, cable inclination, and others could not be considered. Consequently, the predicted maximum modal damping ratios were found to be larger than the measured results [5, 8] and the computation effort was tremendous.

Recently, the writers developed an analytical/numerical hybrid method for studying mitigation of in-plane vibration of sag cables in cable-stayed bridges using oil dampers [9, 10]. This method is able to take account of cable sag, cable inclination, cable internal damping, damper stiffness, damper direction and others, and requires reasonable

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computation effort. The results obtained reveal that the cable sag, damper stiffness and damper direction may affect the performance of oil damper in reducing cable vibration. In particular, when the cable sag parameter λ^2 (the ratio of the elastic-to-catenary stiffness) falls in a certain range around one of the frequency avoidances of inclined cables, the oil damper installed near the cable support will not be able to provide sufficient damping ratio for the corresponding mode of vibration.

In this paper, the hybrid method is further developed to study three-dimensional free and forced vibration of sag cables in cable-stayed bridges with multiple pairs of oil dampers. This is because for most cable-stayed bridges, oil dampers are installed in pairs to stay cables to keep damper systems stable and to reduce both in-plane and out-of-plane cable vibrations. It is, however, unknown as to how to determine the achievable maximum modal damping ratios in both in-plane and out-of-plane vibrational modes of a stayed cable and the optimum parameters for oil dampers. It is also unclear when the results from in-plane vibration studies can be applied to three-dimensional vibration problems. The formulation of the three-dimensional problem concerned is thus presented in part I of this paper. The application and the design-concerned matters are introduced in part II of this paper.

2. BASIC EQUATIONS

2.1. EQUATIONS OF MOTION

This study concerns three-dimensional vibration of an inclined sag cable with multi-pairs of oil dampers installed near the cable support (see Figure 1). The uniform cable is assumed to have small amplitude vibration with respect to its static equilibrium position. By setting the x- and y-co-ordinates in the static profile plane of the cable and taking the left support of the cable as the origin of the Cartesian co-ordinate system, the motion of the cable with oil dampers can be expressed by the three partial differential equations:

$$\frac{\partial}{\partial s}\left[(T+\tau)\left(\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial u}{\partial s}\right)\right]+F_x+\sum_{j=1}^{M} f_{x,j}\,\delta(s-s_{c,j})=m\frac{\partial^2 u}{\partial t^2}+c_1\frac{\partial u}{\partial t} \tag{1}$$

$$\frac{\partial}{\partial s}\left[(T+\tau)\left(\frac{\mathrm{d}y}{\mathrm{d}s}+\frac{\partial w}{\partial s}\right)\right]+F_y+\sum_{j=1}^{M} f_{y,j}\,\delta(s-s_{c,j})=m\frac{\partial^2 w}{\partial t^2}+c_1\frac{\partial w}{\partial t}-mg \tag{2}$$

$$\frac{\partial}{\partial s}\left[(T+\tau)\frac{\partial v}{\partial s}\right]+F_z+\sum_{j=1}^{M} f_{z,j}\,\delta(s-s_{c,j})=m\frac{\partial^2 v}{\partial t^2}+c_2\frac{\partial v}{\partial t} \tag{3}$$

where T is the static cable tension; τ is the dynamic cable tension; u, w and v are the cable dynamic displacement components in the x-, y- and z-directions, respectively, measured from the position of static equilibrium of the cable; s is the Lagrangian co-ordinate in the unstrained cable profile; F_x, F_y and F_z are external dynamic loading per unit length in the x-, y- and z- directions, respectively; $f_{x,j}$, $f_{y,j}$ and $f_{z,j}$ are the forces exerted by the jth pair of oil dampers on the cable at the location of $s_{c,j}$ in the x-, y- and z-directions, respectively; $s_{c,j}$ is the Lagrangian co-ordinate of the jth pair of dampers measured from the left support of the cable; M is the total number of pair of oil dampers; δ is the Dirac's delta function; m is the mass of the cable per unit length; t is the time; c_1 and c_2 are the in-plane and out-of-plane internal damping coefficients of the cable, respectively; and g is the acceleration due to gravity.

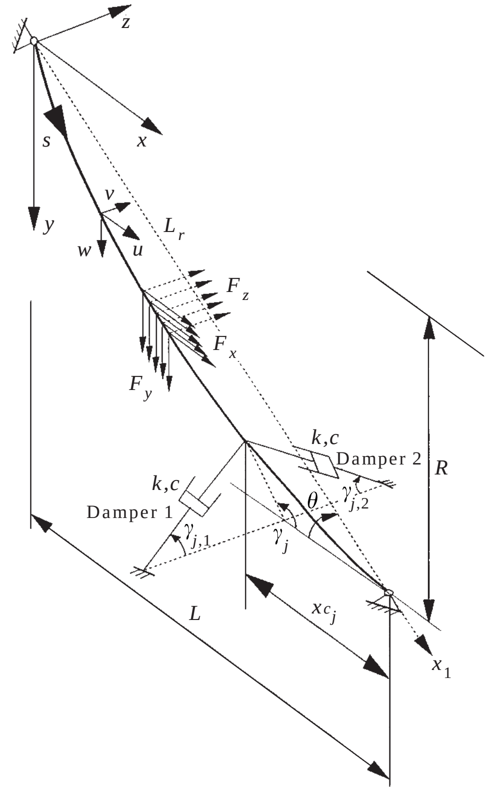

Figure 1. Schematic diagram of an inclined sag cable with oil dampers.

Introducing the following transformations

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{1 + y_x^2}} \frac{\partial}{\partial x} \tag{4}$$

$$H = T \frac{\mathrm{d}x}{\mathrm{d}s} \tag{5}$$

$$h = \tau \frac{\mathrm{d}x}{\mathrm{d}s} \tag{6}$$

and considering the equations of static equilibrium of the cable, equations (1)–(3) can be rewritten as

$$\frac{1}{\sqrt{1 + y_x^2}} \frac{\partial}{\partial x} \left[(H + h) \left(1 + \frac{\partial u}{\partial x} \right) \right] + \sum_{j=1}^{M} f_{x,j}\, \delta(x - (L - x_{c,j})) + F_x = m \frac{\partial^2 u}{\partial t^2} + c_1 \frac{\partial u}{\partial t} \tag{7}$$

$$\frac{1}{\sqrt{1 + y_x^2}} \frac{\partial}{\partial x} \left[(H + h) \frac{\partial w}{\partial x} + h y_x \right] + \sum_{j=1}^{M} f_{y,j}\, \delta(x - (L - x_{c,j})) + F_y = m \frac{\partial^2 w}{\partial t^2} + c_1 \frac{\partial w}{\partial t} \tag{8}$$

$$\frac{1}{\sqrt{1 + y_x^2}} \frac{\partial}{\partial x} \left[(H + h) \frac{\partial v}{\partial x} \right] + \sum_{j=1}^{M} f_{z,j}\, \delta(x - (L - x_{c,j})) + F_z = m \frac{\partial^2 v}{\partial t^2} + c_2 \frac{\partial v}{\partial t} \tag{9}$$

where H is the horizontal component of the static cable tension in the x–y plane; h is the horizontal component of the dynamic cable tension; y_x is the first derivative of y with respect to x; $x_{c,j}$ is the co-ordinate of the jth pair of dampers measured from the right support of the cable; and L is the horizontal length between two cable end-supports in the x–y plane.

The relationship between the dynamic cable tension and dynamic cable displacement in the Lagrangian co-ordinate can be expressed as

$$\tau = EA\,\frac{d\bar{s}^2 - ds^2}{2ds^2} \tag{10}$$

where E is the cable modulus of elasticity; A is the cross-sectional area of the cable; $d\bar{s}$ and ds are the arc-lengths of the deformed and undeformed cable segments, referring to the static cable profile and dynamic cable profile, respectively, through the relations

$$ds^2 = dx^2 + dy^2 \tag{11}$$

$$d\bar{s}^2 = (dx + \partial u)^2 + (dy + \partial w)^2 + \partial v^2. \tag{12}$$

As a result, the horizontal dynamic cable tension h can be expressed as

$$h = \frac{EA}{(1+y_x^2)^{3/2}}\left\{\frac{\partial u}{\partial x} + y_x\frac{\partial w}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\right\}. \tag{13}$$

Since small-amplitude vibration is concerned in this study, the horizontal dynamic cable tension h is reduced to its first order, leading to

$$h = \frac{EA}{(1+y_x^2)^{3/2}}\left[\frac{\partial u}{\partial x} + y_x\frac{\partial w}{\partial x}\right]. \tag{14}$$

Substituting equation (14) into equations (7)–(9) and discarding the differentials of high orders result in

$$\frac{1}{\sqrt{1+y_x^2}}\frac{\partial}{\partial x}\left[\left(H + \frac{EA}{(1+y_x^2)^{3/2}}\right)\frac{\partial u}{\partial x} + \frac{EAy_x}{(1+y_x^2)^{3/2}}\frac{\partial w}{dx}\right] + \sum_{j=1}^{M} f_{x,j}\,\delta(x-(L-x_{c,j})) + F_x = m\frac{\partial^2 u}{\partial t^2} + c_1\frac{\partial u}{\partial t} \tag{15}$$

$$\frac{1}{\sqrt{1+y_x^2}}\frac{\partial}{\partial x}\left[\left(H + \frac{EAy_x^2}{(1+y_x^2)^{3/2}}\right)\frac{\partial w}{\partial x} + \frac{EAy_x}{(1+y_x^2)^{3/2}}\frac{\partial u}{\partial x}\right] + \sum_{j=1}^{M} f_{y,j}\,\delta(x-(L-x_{c,j})) + F_y = m\frac{\partial^2 w}{\partial t^2} + c_1\frac{\partial w}{\partial t} \tag{16}$$

$$\frac{1}{\sqrt{1+y_x^2}}\frac{\partial}{\partial x}\left[H\frac{\partial v}{\partial x}\right] + \sum_{j=1}^{M} f_{z,j}\,\delta(x-(L-x_{c,j})) + F_z = m\frac{\partial^2 v}{\partial t^2} + c_2\frac{\partial v}{\partial t}. \tag{17}$$

The boundary conditions of the cable considered here are

$$u(0,t) = u(L,t) = w(0,t) = w(L,t) = v(0,t) = v(L,t) = 0. \tag{18}$$

2.2. DAMPER FORCES

Let us consider the forces $f_{x,j}$, $f_{y,j}$ and $f_{z,j}$ generated by the jth pair of oil dampers on the cable in the x-, y- and z-directions, respectively. Assume that the pair of oil dampers have the same stiffness k and damping coefficient c. The direction of each damper from the deck to the cable is defined as the positive direction. The direction cosines of damper 1 in the jth pair are denoted by $\cos\alpha_{1,j}$, $\cos\beta_{1,j}$ and $\cos\gamma_{1,j}$, respectively; the direction cosines of damper 2 are $\cos\alpha_{2,j}$, $\cos\beta_{2,j}$ and $\cos\gamma_{2,j}$, respectively. The forces $f_{x,j}$, $f_{y,j}$ and $f_{z,j}$, which depend on the displacement and velocity of the cable at the location $x_{c,j}$ of the jth pair of dampers, can then be determined by the following equation:

$$\begin{bmatrix} f_{x,j} \\ f_{y,j} \\ f_{z,j} \end{bmatrix} = \left\{ \begin{bmatrix} -\cos^2\alpha_{1,j} & -\cos\beta_{1,j}\cos\alpha_{1,j} & -\cos\gamma_{1,j}\cos\alpha_{1,j} \\ -\cos\alpha_{1,j}\cos\beta_{1,j} & -\cos^2\beta_{1,j} & -\cos\gamma_{1,j}\cos\beta_{1,j} \\ -\cos\gamma_{1,j}\cos\alpha_{1,j} & -\cos\gamma_{1,j}\cos\beta_{1,j} & -\cos^2\gamma_{1,j} \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} -\cos^2\alpha_{2,j} & -\cos\beta_{2,j}\cos\alpha_{2,j} & -\cos\gamma_{2,j}\cos\alpha_{2,j} \\ -\cos\alpha_{2,j}\cos\beta_{2,j} & -\cos^2\beta_{2,j} & -\cos\gamma_{2,j}\cos\beta_{2,j} \\ -\cos\gamma_{2,j}\cos\alpha_{2,j} & -\cos\gamma_{2,j}\cos\beta_{2,j} & -\cos^2\gamma_{2,j} \end{bmatrix} \right\}$$

$$\times \begin{bmatrix} ku(L - x_{c,j}, t) + c\dfrac{\partial u(L - x_{c,j}, t)}{\partial t} \\ kw(L - x_{c,j}, t) + c\dfrac{\partial w(L - x_{c,j}, t)}{\partial t} \\ kv(L - x_{c,j}, t) + c\dfrac{\partial v(L - x_{c,j}, t)}{\partial t} \end{bmatrix} \quad (j = 1, 2, \ldots, M). \tag{19}$$

If the jth pair of dampers are installed symmetric to the static equilibrium position of the cable, all the direction cosines can be expressed in terms of two independent variables α_j and γ_j as shown in Figure 1.

$$\cos\alpha_{1,j} = \cos\alpha_{2,j} = \sin\gamma_j\cos\alpha_j \tag{20}$$

$$\cos\beta_{1,j} = \cos\beta_{2,j} = -\sin\gamma_j\sin\alpha_j \tag{21}$$

$$\gamma_{1,j} = 180^\circ - \gamma_{2,j} = \gamma_j. \tag{22}$$

Equation (19) can be then simplified as

$$\begin{bmatrix} f_{x,j} \\ f_{y,j} \\ f_{z,j} \end{bmatrix} = \begin{bmatrix} -2\sin^2\gamma_j\cos^2\alpha_j & 2\sin^2\gamma_j\sin\alpha_j\cos\alpha_j & 0 \\ 2\sin^2\gamma_j\sin\alpha_j\cos\alpha_j & -2\sin^2\gamma_j\sin^2\alpha_j & 0 \\ 0 & 0 & -2\cos^2\gamma_j \end{bmatrix}$$

$$\times \begin{bmatrix} ku(L - x_{c,j}, t) + c\dfrac{\partial u(L - x_{c,j}, t)}{\partial t} \\ kw(L - x_{c,j}, t) + c\dfrac{\partial w(L - x_{c,j}, t)}{\partial t} \\ kv(L - x_{c,j}, t) + c\dfrac{\partial v(L - x_{c,j}, t)}{\partial t} \end{bmatrix} \quad (j = 1, 2, \ldots, M). \tag{23}$$

Clearly, if all pairs of oil dampers are installed to the cable symmetrically, the in-plane damper force components are independent of the out-of-plane damper force components. This is because the in-plane damper force components from the pair of oil dampers caused by the out-of-plane displacement component always cancel each other. The three-dimensional vibration of the cable with oil dampers can thus be decomposed into the mutual independent in-plane vibration and the out-of-plane vibrations according to equations (15)–(17).

2.3. CABLE STATIC PROFILE

The static profile of an inclined cable considering elastic extension can be expressed as [11]:

$$x(s) = \frac{Hs}{EA} + \frac{H}{mg}\left[\sinh^{-1}\left(\frac{V}{H}\right) - \sinh^{-1}\left(\frac{V-mgs}{H}\right)\right] \tag{24}$$

$$y(s) = \frac{mgL_0\,s}{EA}\left(\frac{V}{mgL_0} - \frac{s}{2L_0}\right) + \frac{H}{mg}\left\{\left[1+\left(\frac{V}{H}\right)^2\right]^{1/2} - \left[1+\left(\frac{V-mgs}{H}\right)^2\right]^{1/2}\right\} \tag{25}$$

whre L_0 is the unstrained arc length of the cable; V is the vertical component of the static tension of the cable at the left support in the x–y plane. The first derivative of the static cable displacement y with respect to x is therefore calculated by

$$y_x = \frac{\mathrm{d}y/\mathrm{d}s}{\mathrm{d}x/\mathrm{d}s} = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{26}$$

Equations (24) and (25) must satisfy the two boundary conditions

$$x(L_0) = L \tag{27}$$

$$y(L_0) = R \tag{28}$$

where R is the vertical distance between two supports of the cable in the x–y plane, as shown in Figure 1.

By using an iteration procedure incorporating Newton's method and the dimensionless technique [12] and the two boundary conditions, $\mathrm{d}y/\mathrm{d}x$ can be determined with a quick convergence and a little computational effect.

3. HYBRID METHOD

The hybrid method developed by the writers for both free and forced in-plane vibrations of an inclined sag cable with an oil damper [10] are now extended to the three-dimensional free and forced vibrations of an inclined sag cable with multiple oil dampers. There are three major steps involved in the hybrid method. The first step is to discretize the sag cable into a series of small segments and let the dampers locate at some of the element nodes so that the equations of motion of the cable without dampers can be applied to each segment. The second step is to use an orthogonal transformation to decouple the equations of in-plane motion of the cable and to find the local solutions for each segment. The last step is to assemble these local solutions to form a system matrix by considering the connective conditions between any two segments and using a transfer matrix procedure. From the system matrix in conjunction with the boundary conditions of the cable, the complex eigenvalue and dynamic response of the cable can be finally determined numerically. The following is a summary of the formulae used for three-dimensional free and forced vibrations of an inclined sag cable subject to uniformly distributed harmonic loads with multi-pairs of oil dampers.

3.1. DISCRETIZATION OF THE CABLE

Divide the inclined sag cable into N segments and let the dampers be located at some of the segment nodes (see Figure 2). By assuming y_x to be a constant for each small segment, equations (15)–(17) for the ith segment can be rewritten as

$$\left(H + \frac{\tilde{k}_i}{1 + y_{i,x}^2}\right)\frac{\partial^2 u_i}{\partial x^2} + \frac{\tilde{k}_i\, y_{i,x}}{1 + y_{i,x}^2}\frac{\partial^2 w_i}{\partial x^2} + \tilde{F}_{x,i} = \tilde{m}_i \frac{\partial^2 u_i}{\partial t^2} + \tilde{c}_{1,i}\frac{\partial u_i}{\partial t} \tag{29}$$

$$\left(H + \frac{\tilde{k}_i\, y_{i,x}^2}{1 + y_{i,x}^2}\right)\frac{\partial^2 w_i}{\partial x^2} + \frac{\tilde{k}_i\, y_{i,x}}{1 + y_{i,x}^2}\frac{\partial^2 u_i}{\partial x^2} + \tilde{F}_{y,i} = \tilde{m}_i \frac{\partial^2 w_i}{\partial t^2} + \tilde{c}_{1,i}\frac{\partial w_i}{\partial t} \tag{30}$$

$$H\frac{\partial^2 v_i}{\partial t^2} + \tilde{F}_{z,i} = \tilde{m}_i \frac{\partial^2 v_i}{\partial t^2} + \tilde{c}_{2,i}\frac{\partial v_i}{\partial t} \tag{31}$$

in which

$$y_{i,x} = \frac{\mathrm{d}y_i\,(x)}{\mathrm{d}x} \approx \frac{y_x\,(x_i) + y_x\,(x_{i+1})}{2} \tag{32}$$

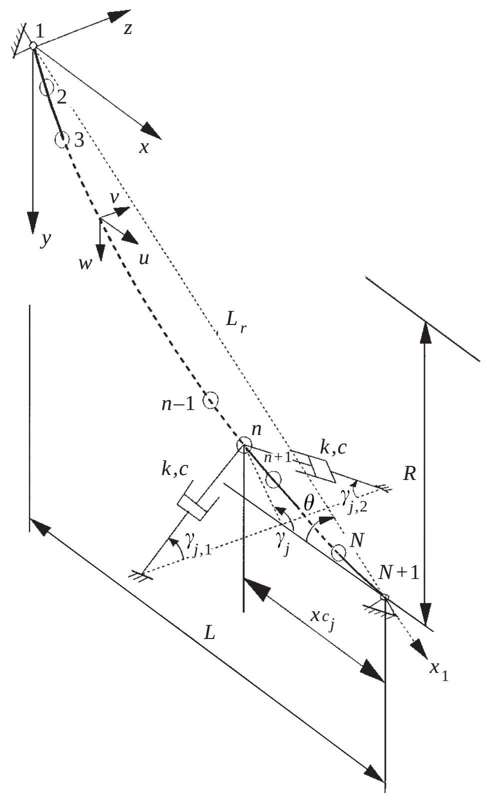

Figure 2. Discretization of cable-damper system.

$$\tilde{k}_i = \frac{EA}{\sqrt{1 + y_{i,x}^2}} \tag{33}$$

$$\tilde{F}_{x,i} = \sqrt{1 + y_{i,x}^2} F_x, \qquad \tilde{F}_{y,i} = \sqrt{1 + y_{i,x}^2} F_y, \qquad \tilde{F}_{z,i} = \sqrt{1 + y_{i,x}^2} F_z \tag{34}$$

$$\tilde{m}_i = \sqrt{1 + y_{i,x}^2}\, m \tag{35}$$

$$\tilde{c}_{1,i} = \sqrt{1 + y_{i,x}^2}\, c_1, \qquad \tilde{c}_{2,i} = \sqrt{y_{i,x}^2}\, c_2. \tag{36}$$

In the above equations, the subscript i indicates the ith segment except for the x-co-ordinate in which the subscript i indicates the ith node. x_i and x_{i+1} are the x-co-ordinates of the left and right side nodes (i.e. the ith and $i+1$th nodes) of the ith segment. For free vibration the external loads $\tilde{F}_{x,i}$, $\tilde{F}_{y,i}$ and $\tilde{F}_{z,i}$ should be dropped.

The solutions $u_i(x, t)$, $w_i(x, t)$ and $v_i(x, t)$ in equations (29)–(31) should satisfy the connective conditions at each node between two segments. The connective conditions are regarded as the continuity of displacement and the equilibrium of force. They are expressed differently according to whether the node is supported by the dampers or not.

If the ith node is not the node where the dampers are installed, the connective conditions are

$$u_{i-1}(x_i, t) = u_i(x_i, t) \tag{37}$$

$$w_{i-1}(x_i, t) = w_i(x_i, t) \tag{38}$$

$$v_{i-1}(x, t) = v_i(x_i, t) \tag{39}$$

$$\begin{aligned} &\left(H + \frac{\tilde{k}_{i-1}}{1 + y_{i-1,x}^2}\right) u_{i-1,x}(x_i, t) + \frac{\tilde{k}_{i-1}\, y_{i-1,x}}{1 + y_{i-1,x}^2} w_{i-1,x}(x_i, t) \\ &\quad = \left(H + \frac{\tilde{k}_i}{1 + y_{i,x}^2}\right) u_{i,x}(x_i, t) + \frac{\tilde{k}_i\, y_{i,x}}{1 + y_{i,x}^2} w_{i,x}(x_i, t) \end{aligned} \tag{40}$$

$$\begin{aligned} &\left(H + \frac{\tilde{k}_{i-1}\, y_{i-1,x}^2}{1 + y_{i-1,x}^2}\right) w_{i-1,x}(x_i, t) + \frac{\tilde{k}_{i-1}\, y_{i-1,x}}{1 + y_{i-1,x}^2} u_{i-1,x}(x_i, t) \\ &\quad = \left(H + \frac{\tilde{k}_i\, y_{i,x}^2}{1 + y_{i,x}^2}\right) w_{i,x}(x_i, t) + \frac{\tilde{k}_i\, y_{i,x}}{1 + y_{i,x}^2} u_{i,x}(x_i, t) \end{aligned} \tag{41}$$

$$H v_{i-1,x}(x_i, t) = H v_{i,x}(x_i, t). \tag{42}$$

If the nth node is the node where the jth pair of dampers are installed, the continuity conditions of displacement remain the same as those expressed by equations (37)–(39). The equilibrium conditions of force at the nth node, however, should be changed to

$$\begin{aligned} &\left(H + \frac{\tilde{k}_{n-1}}{1 + y_{n-1,x}^2}\right) u_{n-1,x}(x_n, t) + \frac{\tilde{k}_{n-1}\, y_{n-1,x}}{1 + y_{n-1,x}^2} w_{n-1,x}(x_n, t) - f_{x,j} \\ &\quad = \left(H + \frac{\tilde{k}_n}{1 + y_{n,x}^2}\right) u_{n,x}(x_n, t) + \frac{\tilde{k}_n\, y_{n,x}}{1 + y_{n,x}^2} w_{n,x}(x_n, t) \end{aligned} \tag{43}$$

$$\left(H+\frac{\tilde{k}_{n-1}\,y_{n-1,x}^{2}}{1+y_{n-1,x}^{2}}\right)w_{n-1,x}\,(x_n,\,t)+\frac{\tilde{k}_{n-1}\,y_{n-1,x}}{1+y_{n-1,x}^{2}}\,u_{n-1,x}\,(x_n,\,t)-f_{y,j}$$

$$=\left(H+\frac{\tilde{k}_{n}\,y_{n,x}^{2}}{1+y_{n,x}^{2}}\right)w_{n,x}\,(x_n,\,t)+\frac{\tilde{k}_{n}\,y_{n,x}}{1+y_{n,x}^{2}}\,u_{n,x}\,(x_n,\,t) \tag{44}$$

$$Hv_{n-1,x}\,(x_n,\,t)-f_{z,j}=Hv_{n,x}\,(x_n,\,t). \tag{45}$$

Correspondingly, the boundary conditions can be rewritten as

$$u_1\,(x_1,\,t)=u_N\,(x_{N+1},\,t)=w_1\,(x_1,\,t)=w_N\,(x_{N+1},\,t)=v_1\,(x_1,\,t)=v_N\,(x_{N+1},\,t)=0. \tag{46}$$

3.2. ORTHOGONAL TRANSFORMATION AND LOCAL SOLUTION

In the case of forced vibration, only uniformly distributed harmonic loads along the cable in x-, y- and z-directions are considered.

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}=\begin{bmatrix} F_x^* \\ F_y^* \\ F_z^* \end{bmatrix}\mathrm{e}^{\Omega t} \tag{47}$$

in which F_x^*, F_y^*, F_z^* are the amplitudes of the harmonic load per length in the x-, y- and z-directions, respectively; and Ω is equal to $I\omega$, in which I is equal to $\sqrt{-1}$ and ω is the frequency of the harmonic load. To be consistent with equations (29)–(31), the external harmonic load is rewritten as

$$\begin{bmatrix} \tilde{F}_{x,i} \\ \tilde{F}_{y,i} \\ \tilde{F}_{z,i} \end{bmatrix}=\sqrt{1+y_{i,x}^{2}}\begin{bmatrix} F_x^* \\ F_y^* \\ F_z^* \end{bmatrix}\mathrm{e}^{\Omega t}=\begin{bmatrix} \tilde{F}_{x,i}^* \\ \tilde{F}_{y,i}^* \\ \tilde{F}_{z,i}^* \end{bmatrix}\mathrm{e}^{\Omega t}. \tag{48}$$

To determine the dynamic response or the modal damping ratio of the cable with discrete oil dampers, a complex eigenvalue analysis should be carried out. By using the method of separation of variables, the solutions of equations (29)–(31) can be written as

$$\begin{bmatrix} u_i\,(x,\,t) \\ w_i\,(x,\,t) \\ v_i\,(x,\,t) \end{bmatrix}=\begin{bmatrix} \phi_i\,(x) \\ \varphi_i\,(x) \\ \psi_i\,(x) \end{bmatrix}\mathrm{e}^{\Omega t} \tag{49}$$

where $\phi_i\,(x)$, $\varphi_i\,(x)$ and $\psi_i\,(x)$ are the complex eigenfunction components of the ith segment in the x-, y- and z-directions. If free vibration is concerned, Ω in equation (49) is taken as the complex eigenvalue given in terms of the modal damping ratio ξ and the pseudo-undamped natural frequency ω of the cable-damper system, that is

$$\Omega=\Omega_r+I\Omega_I=\omega(-\xi\pm I\sqrt{1-\xi^2}) \tag{50}$$

where Ω_r and Ω_I are the real part and imaginary part of the eigenvalues Ω. Substituting equations (48) and (49) into equations (29)–(31) and introducing the following orthogonal coordinate transformation

$$\begin{bmatrix} \phi_i\,(x) \\ \varphi_i\,(x) \\ \psi_i\,(x) \end{bmatrix}=\begin{bmatrix} 1 & y_{i,x} & 0 \\ y_{i,x} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{\phi}_i\,(x) \\ \tilde{\varphi}_i\,(x) \\ \tilde{\psi}_i\,(x) \end{bmatrix} \tag{51}$$

a set of uncoupled equations of motion for the ith cable segment can be obtained

$$(H+\tilde{k}_i)\tilde{\phi}_{i,xx}-(\tilde{m}_i\,\Omega^2+\tilde{c}_{1,i}\,\Omega)\tilde{\phi}_i+\frac{\tilde{F}^*_{x,i}+y_{i,x}\,\tilde{F}^*_{y,i}}{1+y^2_{i,x}}=0 \tag{52}$$

$$H\tilde{\varphi}_{i,xx}-(\tilde{m}_i\,\Omega^2+\tilde{c}_{1,i}\,\Omega)\tilde{\varphi}_i+\frac{y_{i,x}\,\tilde{F}^*_{x,i}-\tilde{F}^*_{y,i}}{1+y^2_{i,x}}=0 \tag{53}$$

$$\tilde{H}\tilde{\psi}_{i,xx}-(\tilde{m}_i\,\Omega^2+\tilde{c}_{2,i}\,\Omega)\tilde{\psi}_i+\tilde{F}^*_{z,i}=0. \tag{54}$$

The local solutions for these equations then are

$$\tilde{\phi}_i\,(x)=D_{1i}\,\mathrm{e}^{(r_{1i}+Ir_{2i})x}+D_{2i}\,\mathrm{e}^{-(r_{1i}+Ir_{2i})x}+\frac{1}{m\Omega^2+\Omega c_1}\frac{F^*_x+y_{i,x}\,F^*_y}{1+y^2_{i,x}} \tag{55}$$

$$\tilde{\varphi}_i\,(x)=D_{3i}\,\mathrm{e}^{(r_{3i}+Ir_{4i})x}+D_{4i}\,\mathrm{e}^{-(r_{3i}+Ir_{4i})x}+\frac{1}{m\Omega^2+\Omega c_1}\frac{y_{i,x}\,F^*_x-F^*_y}{1+y^2_{i,x}} \tag{56}$$

$$\tilde{\psi}_i\,(x)=D_{5i}\,\mathrm{e}^{(r_{5i}+Ir_{6i})x}+D_{6i}\,\mathrm{e}^{-(r_{5i}+Ir_{6i})x}+\frac{F^*_z}{m\Omega^2+\Omega c_2}. \tag{57}$$

For the free vibration, D_{ji} ($j=1,2,3,4,5,6;\ i=1,2,\ldots,N$) are the unknown complex constants associated with the ith cable segment. Expressions for the parameters r_{ji} ($j=1,2,3,4,5,6;\ i=1,2,\ldots,N$) are as follows:

when $(2m\Omega_r+c_1)\geqslant 0$

$$r_{1i}=\sqrt{\frac{-\lambda_{1ir}+\sqrt{\lambda^2_{1ir}+\lambda^2_{1iI}}}{2}},\qquad r_{2i}=\sqrt{\frac{\lambda_{1ir}+\sqrt{\lambda^2_{1ir}+\lambda^2_{1iI}}}{2}}, \tag{58}$$

$$r_{3i}=\sqrt{\frac{-\lambda_{2ir}+\sqrt{\lambda^2_{2ir}+\lambda^2_{2iI}}}{2}},\qquad r_{4i}=\sqrt{\frac{\lambda_{2ir}+\sqrt{\lambda^2_{2ir}+\lambda^2_{2iI}}}{2}}, \tag{59}$$

when $(2m\Omega_r+c_1)<0$

$$r_{1i}=\sqrt{\frac{-\lambda_{1ir}+\sqrt{\lambda^2_{1ir}+\lambda^2_{1iI}}}{2}},\qquad r_{2i}=-\sqrt{\frac{\lambda_{1ir}+\sqrt{\lambda^2_{1ir}+\lambda^2_{1iI}}}{2}}, \tag{60}$$

$$r_{3i}=\sqrt{\frac{-\lambda_{2ir}+\sqrt{\lambda^2_{2ir}+\lambda^2_{2iI}}}{2}},\qquad r_{4i}=-\sqrt{\frac{\lambda_{2ir}+\sqrt{\lambda^2_{2ir}+\lambda^2_{2iI}}}{2}}, \tag{61}$$

and, when $(2m\Omega_r+c_2)\geqslant 0$

$$r_{5i}=\sqrt{\frac{-\lambda_{3ir}+\sqrt{\lambda^2_{3ir}+\lambda^2_{3iI}}}{2}},\qquad r_{6i}=\sqrt{\frac{\lambda_{3ir}+\sqrt{\lambda^2_{3ir}+\lambda^2_{3iI}}}{2}}, \tag{62}$$

when $(2m\Omega_r+c_2)<0$

$$r_{5i}=\sqrt{\frac{-\lambda_{3ir}+\sqrt{\lambda^2_{3ir}+\lambda^2_{3iI}}}{2}},\qquad r_{6i}=\sqrt{\frac{\lambda_{3ir}+\sqrt{\lambda^2_{3ir}+\lambda^2_{3iI}}}{2}}. \tag{63}$$

In the above equations,

$$\lambda_{1ir} = \frac{\tilde{m}_i\,(\Omega_1^2 - \Omega_r^2) - \tilde{c}_{1,i}\,\Omega_r}{H + \tilde{k}_i}, \qquad \lambda_{1iI} = \frac{2\tilde{m}_i\,\Omega_r + \tilde{c}_{1,i}}{H + \tilde{k}_i}\,\Omega_I \tag{64}$$

$$\lambda_{2ir} = \frac{\tilde{m}_i\,(\Omega_I^2 - \Omega_r^2) - \tilde{c}_{1,i}\,\Omega_r}{H}, \qquad \lambda_{2iI} = \frac{2\tilde{m}_i\,\Omega_r + \tilde{c}_{1,i}}{H}\,\Omega_I \tag{65}$$

$$\lambda_{3ir} = \frac{\tilde{m}_i\,(\Omega_I^2 - \Omega_r^2) - \tilde{c}_{2,i}\,\Omega_r}{H}, \qquad \lambda_{3iI} = \frac{2\tilde{m}_i\,\Omega_r + \tilde{c}_{2,i}}{H}\,\Omega_I. \tag{66}$$

For forced vibration, the above equations are still valid if Ω_r is set zero.

3.3. TRANSFER MATRIX AND GLOBAL SOLUTION

By substituting the local solutions, equations (55)–(57), into the orthogonal transformation equation (51), and then into the connective conditions, equations (37)–(45), the relationship of the six complex constants between the two neighbouring segments can be established in a matrix form.

$$\begin{bmatrix} D_{1i} \\ D_{2i} \\ D_{3i} \\ D_{4i} \\ D_{5i} \\ D_{6i} \end{bmatrix} = [T]^i_{6\times 6}\,(x_i) \begin{bmatrix} D_{1i-1} \\ D_{2i-1} \\ D_{3i-1} \\ D_{4i-1} \\ D_{5i-1} \\ D_{6i-1} \end{bmatrix}. \tag{67}$$

The matrix $[T]^i_{6\times 6}\,(x_i)$ is called the ith transfer matrix which transfers the six complex constants of the i-1th segment to those of the ith segment. It is a 6×6 matrix in the three-dimensional cable vibration problem, expressed as

$$[T]^i_{6\times 6}\,(x_i) = [\overline{R}]^i_{6\times 6}\,(x_i)\,[R]^{i-1}_{6\times 6}\,(x_i) \tag{68}$$

where $[\overline{R}]^i_{6\times 6}\,(x_i)$ is the inverse matrix of $[R]^i_{6\times 6}\,(x_i)$; $[R]^i_{6\times 6}\,(x_i)$ and $[R]^{i-1}_{6\times 6}\,(x_i)$ are the 6×6 complex matrices for the $i-1$th and ith segments at node i, respectively, of which the elements are given in the Appendix.

Repeated use of equation (67) for all elements results in

$$[R]^N_{6\times 6}\,(x_{N+1}) \begin{bmatrix} D_{1N} \\ D_{2N} \\ D_{3N} \\ D_{4N} \\ D_{5N} \\ D_{6N} \end{bmatrix} = [B]_{6\times 6} \begin{bmatrix} D_{11} \\ D_{21} \\ D_{31} \\ D_{41} \\ D_{41} \\ D_{61} \end{bmatrix} \tag{69}$$

where

$$[B]_{6\times 6} = [R]^N_{6\times 6}\,(x_{N+1})\,[T]^N_{6\times 6}\,(x_N)\cdots[T]^k_{6\times 6}\,(x_k)\,[T]^{k-1}_{6\times 6}\,(x_{k-1})\cdots[T]^2_{6)6}\,(x_2). \tag{70}$$

Satisfaction of the displacement boundary conditions at the nodes 1 and $N+1$ gives the following characteristic equations for eigenvalue problem

$$\det\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} \end{bmatrix} = 0 \tag{71}$$

and the following equation for the forced vibration of the cable

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} \end{bmatrix} \begin{bmatrix} D_{11} \\ D_{21} \\ D_{31} \\ D_{41} \\ D_{51} \\ D_{61} \end{bmatrix} = \begin{bmatrix} \dfrac{F_x^*}{m\omega^2 - I\omega c_1} \\ \dfrac{F_y^*}{m\omega^2 - I\omega c_1} \\ \dfrac{F_x^*}{m\omega^2 - I\omega c_1} \\ \dfrac{F_y^*}{m\omega^2 - I\omega c_1} \\ \dfrac{F_z^*}{m\omega^2 - I\omega c_2} \\ \dfrac{F_z^*}{m\omega^2 - I\omega c_2} \end{bmatrix} \tag{72}$$

in which B_{ij} ($i = 1, 2, 3; j = 1, 2, 3, 4, 5, 6$) are the elements of matrix $[B]_{6\times 6}$.

If the pairs of oil dampers are symmetrically installed to the cable with respect to the static equilibrium position of the cable, the in-plane vibration of the cable-damper system is independent of the out-of-plane vibration, as indicated by equations (15)–(17) and equation (23). Therefore, there exist

$$B_{15} = B_{16} = B_{25} = B_{26} = B_{31} = B_{32} = B_{33} = B_{34} = 0. \tag{73}$$

The in-plane and out-of-plane dynamic responses and modal damping ratios can be determined separately. For instance, two complex equations can be extracted from equation (71).

$$R_1(\Omega_r, \Omega_I) + I \cdot \mathrm{Im}_1(\Omega_r, \Omega_I) = 0 \tag{74}$$

$$R_2(\Omega_r, \Omega_I) + I \cdot \mathrm{Im}_2(\Omega_r, \Omega_I) = 0. \tag{75}$$

Equation (74) corresponds to the in-plane vibration while equation (75) corresponds to the out-of-plane vibration. R_j, Im_j ($j = 1, 2$) are the real part and imaginary part, respectively. From either equation (74) or equation (75), the real part Ω_r and imaginary part Ω_1 of the eigenvalue Ω can be determined for either the in-plane vibration or the out-of-plane vibration. The use of equation (50) then gives the corresponding modal damping ratios and pseudo-undamped natural frequencies of the system. The complex eigenfunction components for the ith element can be obtained in terms of the back substitution of the eigenvalues, and the complex eigenfunction components for the whole cable can then be assembled from all elements.

For the forced vibration, once the first six complex constants at the first node are determined from equation (72), all the other complex constants can be determined using equation (67). Finally, the dynamic response of the cable with oil dampers is determined through equations (55)–(57), equation (51) and then equation (49).

If the pairs of oil dampers are unsymmetrically installed to the cable with respect to the static equilibrium position of the cable, the in-plane vibration of the cable-damper system is coupled with the out-of-plane vibration due to the pairs of oil dampers. The modal damping ratio or the dynamic response of the system can be obtained using equation (71) or equation (72) as a whole.

4. CONCLUSIONS

A formulation for determining both dynamic response due to harmonic loading and modal damping ratio of three-dimensional vibration of an inclined sag cable with multi-pairs of oil dampers have been presented in this paper in terms of a hybrid method. The hybrid method consists of three major steps: cable discretization; orthogonal transformation for local solution; and transfer matrix procedure for global solution. This method can naturally consider cable sag, cable inclination, cable internal damping, damper direction, damper stiffness, and others. Multi-pairs of oil dampers with either symmetric or unsymmetric arrangement can also be dealt with. As seen from the derived formulae, three-dimensional small-amplitude vibration of the cable-damper system can be decomposed into mutual-independent in-plane and out-of-plane vibrations if multi-pairs of oil dampers are symmetrically installed to the cable with respect to its static equilibrium position. Otherwise, the in-plane and out-of-plane vibrations of the cable-damper system are coupled due to the oil dampers installed unsymmetrically. The application of the derived formulae to sag cables in a real cable-stayed bridge will be presented in part II of this paper together with extensive parametric studies. The formulation for determining dynamic response of three dimensional inclined sag cables with oil dampers installed and under real wind excitations, in particular under wind–rain excitation, represents a difficult task and needs further investigation.

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APPENDIX: ELEMENTS OF MATRIXES $[R]^{i-1}_{6\times 6}(x_i)$ and $[R]^{i}_{6\times 6}(x_i)$

If the ith node is not supported by oil dampers, the element R_{ij} ($i = 1, 2, 3, 4, 5, 6$; $j = 1, 2, 3, 4, 5, 6$) in the matrix $[R]^{i-1}_{6\times 6}(x_i)$ are as follows:

$$R_{11} = \mathrm{e}^{(r_{1i-1}+Ir_{2i+1})x_i}, \qquad R_{12} = \mathrm{e}^{-(r_{1i-1}+Ir_{2i-1})x_i}, \qquad R_{13} = y_{i-1,x}\,\mathrm{e}^{(r_{3i-1}+Ir_{4i-1})x_i},$$

$$R_{14} = y_{i-1,x}\,\mathrm{e}^{-(r_{3i-1}+Ir_{4i-1})x_i}, \qquad R_{15} = 0, \qquad R_{16} = 0;$$

$$R_{21} = y_{i-1,x}\,\mathrm{e}^{(r_{1i-1}+Ir_{2i-1})x_i}, \qquad R_{22} = y_{i-1,x}\,\mathrm{e}^{-(r_{1i-1}+Ir_{2i-1})x_i}, \qquad R_{23} = -\mathrm{e}^{(r_{3i-1}+Ir_{4i-1})x_i},$$

$$R_{24} = -\mathrm{e}^{-(r_{3i-1}+Ir_{4i-1})x_i}, \qquad R_{25} = 0, \qquad R_{26} = 0;$$

$$R_{31} = R_{32} = R_{33} = R_{34} = 0, \qquad R_{35} = \mathrm{e}^{(r_{5i-1}+Ir_{6i-1})x_i}, \qquad R_{36} = \mathrm{e}^{-(r_{5i-1}+I_{6i-1})i};$$

$$R_{41} = (H + \tilde{k}_{i-1})\,(r_{1i-1} + Ir_{2i-1})\,\mathrm{e}^{(r_{1i-1}+Ir_{2i-1})x_i},$$

$$R_{42} = -(H + \tilde{k}_{i-1})\,(r_{1i-1} + Ir_{2i-1})\,\mathrm{e}^{-(r_{1i-1}+Ir_{2i-1})x_i},$$

$$R_{43} = Hy_{i-1,x}\,(r_{3i-1} + Ir_{4i-1})\,\mathrm{e}^{(r_{3i-1}+Ir_{4i-1})x_i},$$

$$R_{44} = -Hy_{i-1,x}\,(r_{3i-1} + Ir_{4i-1})\,\mathrm{e}^{-(r_{3i-1}+Ir_{4i-1})x_i}, \qquad R_{45} = 0, \qquad R_{46} = 0;$$

$$R_{51} = (H + \tilde{k}_{i-1})y_{i-1,x}\,(r_{1i-1} + Ir_{2i-1})\,\mathrm{e}^{(r_{1i-1i}+Ir_{2i-1})x_i},$$

$$R_{52} = -(H + \tilde{k}_{i-1})y_{i-1,x}\,(r_{1i-1} + Ir_{2i-1})\,\mathrm{e}^{-(r_{1i-1}+Ir_{2i-1})x_i},$$

$$R_{53} = -H(r_{3i-1} + Ir_{4i-1})\,\mathrm{e}^{(r_{3i-1}+Ir_{4i-1})x_i}, \qquad R_{54} = H(r_{3i-1} + Ir_{4i-1})\,\mathrm{e}^{-(r_{3i-1}+Ir_{4i-1})x_i},$$

$$R_{55} = 0, \qquad R_{56} = 0;$$

$$R_{61} = R_{62} = R_{63} = R_{64} = 0, \qquad R_{65} = H(r_{5i-1} + Ir_{6i-1})\,\mathrm{e}^{(r_{5i-1}+Ir_{6i-1})x_i},$$

$$R_{66} = -H(r_{5i-1} + Ir_{6i-1})\,\mathrm{e}^{-(r_{5i-1}+Ir_{6i-1})x_i}.$$

Replacing r_{1i-1}, r_{2i-1}, r_{3i-1}, r_{4i-1}, r_{5i-1}, r_{6i-1} by r_{1i}, r_{2i}, r_{3i}, r_{4i}, r_{5i}, r_{6i} and $\tilde{k}_{i-1}$, $y_{i-1,x}$ by $\tilde{k}_i$, $y_{i,x}$ in the above elements gives the corresponding elements in the matrix $[R]^{i}_{6\times 6}(x_i)$.

If the ith node is equal to the nth node where the jth pair of dampers are installed, some elements in the matrix $[R]^{i-1}_{6\times 6}(x_i)$ should be changed whereas all elements in the $[R]^{i}_{6\times 6}(x_i)$ remain the same.

The elements R_{41}, R_{42}, R_{43}, R_{44}, R_{45}, R_{46}, R_{51}, R_{52}, R_{53}, R_{54}, R_{55}, R_{56}, R_{61}, R_{62}, R_{63}, R_{64}, R_{65} and R_{66} in the matrix $[R]^{i-1}_{6\times 6}(x_i)$ now should be

$$R_{41} = (H + \tilde{k}_{n-1})\,(r_{1n-1} + Ir_{2n-1})\,\mathrm{e}^{(r_{1n-1}+Ir_{2n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{(r_{1n-1}+Ir_{2n-1})x_n}$$

$$\times [(\cos\alpha_{1,j} + y_{n-1,x}\cos\beta_{1,j})\cos\alpha_{1,j} + (\cos\alpha_{2,j} + y_{n-1,x}\cos\beta_{2,j})\cos\alpha_{2,j}],$$

$$R_{42} = -(H + \tilde{k}_{n-1})(r_{1n-1} + Ir_{2n-1})\,\mathrm{e}^{-(r_{1n-1} + Ir_{2n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{-(r_{1n-1} + Ir_{2n-1})x_n}$$
$$\times[(\cos\alpha_{1,j} + y_{n-1,x}\cos\beta_{1,j})\cos\alpha_{1,j} + (\cos\alpha_{2,j} + y_{n-1,x}\cos\beta_{2,j})\cos\alpha_{2,j}],$$
$$R_{43} = Hy_{n-1,x}(r_{3n-1} + Ir_{4n-1})\,\mathrm{e}^{(r_{3n-1} + Ir_{4n-1})x_n} + (k + \Omega c)e^{(r_{3n-1} + Ir_{4n-1})x_n}$$
$$\times[(y_{n-1,x}\cos\alpha_{1,j} - \cos\beta_{1,j})\cos\alpha_{1,j} + (y_{n-1,x}\cos\alpha_{2,j} - \cos\beta_{2,j})\cos\alpha_{2,j}],$$
$$R_{44} = -Hy_{n-1,x}(r_{3n-1} + Ir_{4n-1})\,\mathrm{e}^{-(r_{3n-1} + Ir_{4n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{-(r_{3n-1} + Ir_{4n-1})x_n}$$
$$\times[y_{n-1,x}\cos\alpha_{1,j} - \cos\beta_{1,j})\cos\alpha_{1,j} + y_{n-1,x}\cos\alpha_{2,j} - \cos\beta_{2,j})\cos\alpha_{2,j}],$$
$$R_{45} = (k + \Omega c)\,\mathrm{e}^{(r_{5n-1} + Ir_{6n-1})x_n}[\cos\gamma_{1,j}\cos\alpha_{1,j} + \cos\gamma_{2,j}\cos\alpha_{2,j}],$$
$$R_{46} = (k + \Omega c)\,\mathrm{e}^{-(r_{5n-1} + Ir_{6n-1})x_n}[\cos\gamma_{1,j}\cos\alpha_{1,j} + \cos\gamma_{2,j}\cos\alpha_{2,j}];$$
$$R_{51} = (H + \tilde{k}_{n-1})y_{n-1,x}(r_{1n-1} + Ir_{2n-1})\,\mathrm{e}^{(r_{1n-1} + Ir_{2n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{(r_{1n-1} + Ir_{2n-1})x_n}$$
$$\times[(\cos\alpha_{1,j} + y_{n-1,x}\cos\beta_{1,j})\cos\beta_{1,j} + (\cos\alpha_{2,j} + y_{n-1,x}\cos\beta_{2,j})\cos\beta_{2,j}],$$
$$R_{52} = -(H + \tilde{k}_{n-1})y_{n-1,x}(r_{1n-1} + Ir_{2n-1})\,\mathrm{e}^{-(r_{1n-1} + Ir_{2n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{-(r_{1n-1} + Ir_{2n-1})x_n}$$
$$\times[(\cos\alpha_{1,j} + y_{n-1,x}\cos\beta_{1,j})\cos\beta_{1,j} + (\cos\alpha_{2,j} + y_{n-1,x}\cos\beta_{2,j})\cos\beta_{2,j}],$$
$$R_{53} = -H(r_{3n-1} + Ir_{4n-1})\,\mathrm{e}^{(r_{3n-1} + Ir_{4n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{(r_{3n-1} + Ir_{4n-1})x_n}$$
$$\times[(y_{n-1,x}\cos\alpha_{1,j} - \cos\beta_{1,j})\cos\beta_{1,j} + (y_{n-1,x}\cos\alpha_{2,j} - \cos\beta_{2,j})\cos\beta_{2,j}],$$
$$R_{54} = H(r_{3n-1} + Ir_{4n-1})\,\mathrm{e}^{-(r_{3n-1} + Ir_{4n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{-(r_{3n-1} + Ir_{4n-1})x_n}$$
$$\times[(y_{n-1,x}\cos\alpha_{1,j} - \cos\beta_{1,j})\cos\beta_{1,j} + (y_{n-1,x}\cos\alpha_{2,j} - \cos\beta_{2,j})\cos\beta_{2,j}],$$
$$R_{55} = (k + \Omega c)\,\mathrm{e}^{(r_{5n-1} + Ir_{6n-1})}[\cos\gamma_{1,j}\cos\beta_{1,j} + \cos\gamma_{2,j}\cos\beta_{2,j}],$$
$$R_{56} = (k + \Omega c)\,\mathrm{e}^{-(r_{5n-1} + Ir_{6n-1})}[\cos\gamma_{1,j}\cos\beta_{1,j} + \cos\gamma_{2,j}\cos\beta_{2,j}];$$
$$R_{61} = (k + \Omega c)\,\mathrm{e}^{(r_{1n-1} + Ir_{2n-1})x_n}$$
$$\times[(\cos\alpha_{1,j} + y_{n-1,x}\cos\beta_{1,j})\cos\gamma_{1,j} + (\cos\alpha_{2,j} + y_{n-1,x}\cos\beta_{2,j})\cos\gamma_{2,j}],$$
$$R_{62} = (k + \Omega c)\,\mathrm{e}^{-(r_{1n-1} + Ir_{2n-1})x_n}$$
$$\times[(\cos\alpha_{1,j} + y_{n-1,x}\cos\beta_{1,j})\cos\gamma_{1,j} + (\cos\alpha_{2,j} + y_{n-1,x}\cos\beta_{2,j})\cos\gamma_{2,j}],$$
$$R_{63} = (k + \Omega c)\,\mathrm{e}^{(r_{3n-1} + Ir_{4n-1})x_n}$$
$$\times[(y_{n-1,x}\cos\alpha_{1,j} - \cos\beta_{1,j})\cos\gamma_{1,j} + (y_{n-1,x}\cos\alpha_{2,j} - \cos\beta_{2,j})\cos\gamma_{2,j}],$$
$$R_{64} = (k + \Omega c)\,\mathrm{e}^{-(r_{3n-1} + Ir_{4n-1})x_n}$$
$$\times[(y_{n-1,x}\cos\alpha_{1,j} - \cos\beta_{1,j})\cos\gamma_{1,j} + (y_{n-1,x}\cos\alpha_{2,j} - \cos\beta_{2,j})\cos\gamma_{2,j}],$$
$$R_{65} = H(r_{5n-1} + Ir_{6n-1})\,\mathrm{e}^{(r_{5n-1} + Ir_{6n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{(r_{5n-1} + Ir_{5n-1})}[\cos^2\gamma_{1,j} + \cos^2\gamma_{2,j}],$$
$$R_{66} = -H(r_{5n-1} + Ir_{6n-1})\,\mathrm{e}^{-(r_{5n-1} + Ir_{6n-1})x_n} + (k + \Omega c)\,\mathrm{e}^{-(r_{5n-1} + Ir_{5n-1})}[\cos^2\gamma_{1,j} + \cos^2\gamma_{2,j}].$$